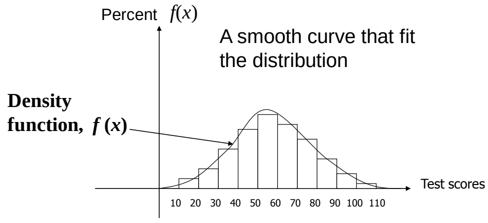
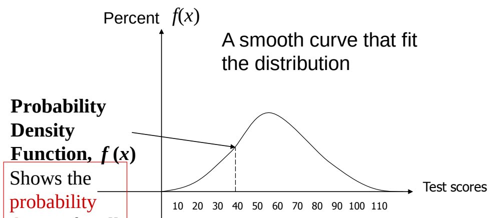
Continuous Distributions

1 Random Variables of the Continuous Type

Density Curve



Density Curve



Shows the probability density for all values of x.

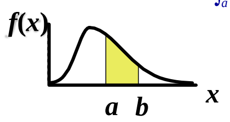
Probability density is not probability!

Use a mathematical model to describe the variable.

Continuous Distribution

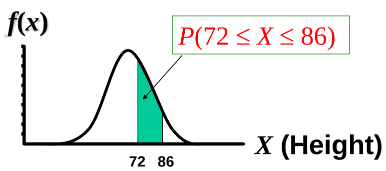
Probability Density Function (*p.d.f.*) of a random variable X of continuous type with a space S is an integrable function, $f(x)$, that satisfying the following conditions:

1. $f(x) \geq 0,\ x \in S$,
2. $\int_S f(x)\,dx = 1$, (Total area under curve is 1.)
3. For a and b in S, $P(a < X < b) = \int_a^b f(x)dx$



Meaning of Area Under Curve

Example: What percentage of the distribution is in between 72 and 86?

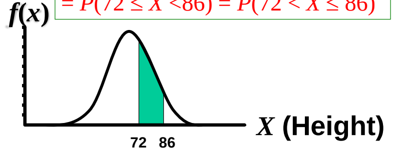


$P(X = 72) = 0$, density is not probability.

Meaning of Area Under Curve

Example: What percentage of the distribution is in between 72 and 86?

$P(72 \leq X \leq 86) = P(72 < X < 86)$
$= P(72 \leq X < 86) = P(72 < X \leq 86)$



Example:

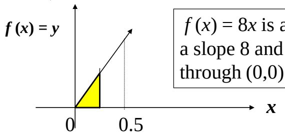
If the density function of a continuous distribution is

$$f(x) = \begin{cases} 8x \;, & \text{for } 0 < x < 0.5 \\ 0 \;, & \text{elsewhere} \end{cases}$$

Find the proportion of values in this distribution that is less than 1/4.

The area under the $f(x)$ between 0 and 1/4 $= \int_0^{1/4} 8x\,dx$

$$= 8\frac{x^2}{2}\Big|_0^{1/4} = 4x^2\Big|_0^{1/4} = 4(1/4)^2 - 4(0)^2 = 4/16 = 1/4.$$



$f(x) = 8x$ is a line with a slope 8 and passes through (0,0)

Review of Calculus

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad \int x^3 dx = \frac{1}{3+1}x^{3+1} + c = \frac{1}{4}x^4 + c$$

$$\int e^{ax}dx = \int e^u \frac{1}{a}du \qquad u = ax,\; du = adx$$

$$= \frac{1}{a}e^u + c \qquad \int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

Example:

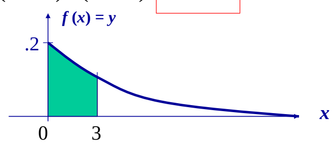
If the density function of a continuous distribution X, waiting time between arrivals of cars at a intersection, is

$$f(x) = \frac{1}{5}e^{-\frac{x}{5}}, \quad \text{for } x > 0$$

Find the probability that the waiting time (in seconds) till the next arrival of car at this intersection is less than 3 seconds.

The area under the $f(x)$ below 3 $= \int_0^3 \frac{1}{5}e^{-\frac{x}{5}}dx$

$$= (-e^{-x/5})\Big|_0^3 = (-e^{-3/5}) - (-e^{-0/5}) = 1 - e^{-3/5} = 1 - 0.5488 = 0.4522$$



Cumulative Distribution Function

The cumulative distribution function (c.d.f. or distribution function, d.f.) of a continuous random variable is defined as

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

- $F(-\infty) = 0, \quad F(\infty) = 1$
- $P(a < X < b) = F(b) - F(a)$
- $F'(x) = f(x)$, if derivative exists

Measure of Center for a Continuous Distribution

The mean value (expected value) of a continuous random variable (distribution) X, denoted by μ_X or just μ (or $E[X]$) is defined as

$$\mu_X = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

Measure of Spread for a Continuous Distribution

The **variance** of a continuous random variable (distribution) X, denoted by σ_X^2 or just σ^2 (or $Var[X]$) is defined as

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx$$

The standard deviation of X is

$$\sigma = \sqrt{\sigma^2} = \sqrt{E[(X-\mu)^2]}$$

Example:

If the density function of a continuous distribution is

$$f(x)=\begin{cases} 8x \ , & \text{for } \ 0<x<0.5 \\ 0 \ , & \text{elsewhere} \end{cases}$$

Find the mean and variance of this distribution

The mean is

$$\mu = E(X)=\int_{-\infty}^{\infty} x\cdot f(x)\,dx=\int_{0}^{0.5} x\cdot f(x)\,dx$$

$$=\int_0^{0.5} x\cdot 8x\,dx = 8\frac{x^3}{3}\Big|_0^{0.5}=\frac{8}{3}(.125-0)=\boxed{\frac{1}{3}}$$

13

Example:

If the density function of a continuous distribution is

$$f(x)=\begin{cases} 8x \ , & \text{for } \ 0<x<0.5 \\ 0 \ , & \text{elsewhere} \end{cases}$$

Find the mean and variance of this distribution

The variance is $\sigma^2 = Var(X)=E[X^2]-(E[X])^2$

$$E[X^2]=\int_{-\infty}^{\infty} x^2\cdot f(x)\,dx=\int_0^{0.5} x^2\cdot f(x)\,dx$$

$$=\int_0^{0.5} x^2\cdot 8x\,dx = 8\frac{x^4}{4}\Big|_0^{0.5}=\frac{8}{4}(.0625-0)=\boxed{\frac{1}{8}}$$

$$\sigma^2 = E[X^2]-(E[X])^2=\frac{1}{8}-\left(\frac{1}{3}\right)^2=\boxed{\frac{1}{72}}$$

$M''(0)$ $\quad M'(0)^2$

14

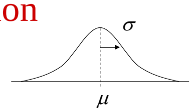
The Normal Distribution

1 The Normal Distribution

15

Normal Probability Density Function

The continuous random variable X has a **normal distribution** if its *p.d.f.* is



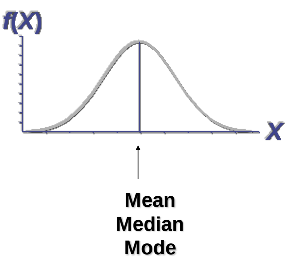
$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty<x<\infty$$

Notation: $\mathcal{N}(\mu,\sigma^2)\Rightarrow$ A normal distribution with mean μ and standard deviation σ

16

Normal Distribution

1. "Bell-Shaped" & Symmetrical
2. Mean, Median, Mode Are Equal
3. Random Variable Has Infinite Range $-\infty < x < \infty$

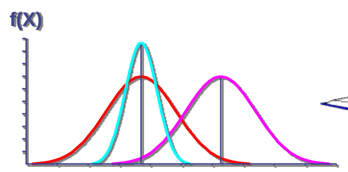


17

Infinite Number of Tables

Normal distributions differ by mean & standard deviation.

Each distribution would require its own table.



That's an *infinite* number!

18

Standard Normal Distribution

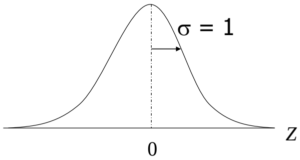
Standard Normal Distribution:

A normal distribution with *mean* = 0 and *standard deviation* = 1.

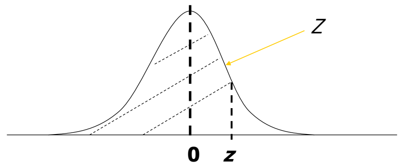
Notation:

$Z \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$

Cap letter **Z**



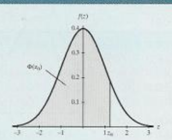
Area under Standard Normal Curve



How to find the proportion of the are under the standard normal curve below z or say P (Z < z) = ?

Use Standard Normal Table!!!

Table Va: The Normal Distribution



$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

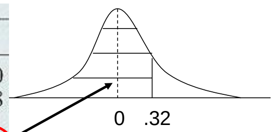
$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

Standard Normal Distribution

$P(Z < 0.32) = \Phi(0.32)$ = Area below .32 = 0.6255

z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
0.3	0.6179	0.6217	0.6255
0.4	0.6554	0.6591	0.6628

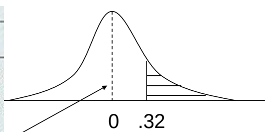


Standard Normal Distribution

$P(Z > 0.32)$ = Area above .32 = 1 – .6255 = .3745

Areas in the upper tail of the standard normal distribution

z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
0.3	0.6179	0.6217	0.6255
0.4	0.6554	0.6591	0.6628

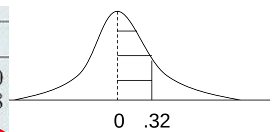


Standard Normal Distribution

$P(0 < Z < 0.32)$ = Area between 0 and .32 = ?

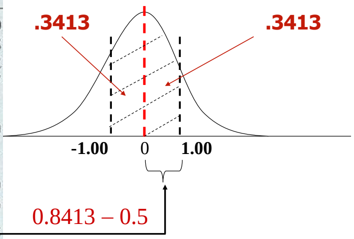
= 0.6255 – 0.5 = **0.1255**

z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
0.3	0.6179	0.6217	0.6255
0.4	0.6554	0.6591	0.6628



P (-1.00 < *Z* < 1.00) = .6826

z	0.00	0.01
0.0	0.5000	0.5040
0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611
0.8	0.7881	0.7910
0.9	0.8159	0.8186
1.0	0.8413	0.8438
1.1	0.8643	0.8665

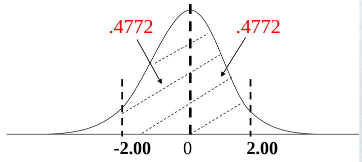


0.8413 – 0.5

25

P (-2.00 < *Z* < 2.00) = .9544

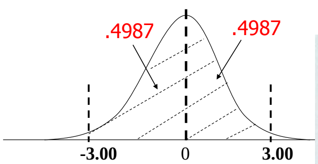
z	0.00
0.0	0.5000
0.1	0.5398
0.2	0.5793
0.3	0.6179
0.4	0.6554
0.5	0.6915
0.6	0.7257
0.7	0.7580
0.8	0.7881
0.9	0.8159
1.0	0.8413
1.1	0.8643
1.2	0.8849
1.3	0.9032
1.4	0.9192
1.5	0.9332
1.6	0.9452
1.7	0.9554
1.8	0.9641
1.9	0.9713
2.0	0.9772
2.1	0.9821



.9544 = .4772 + .4772

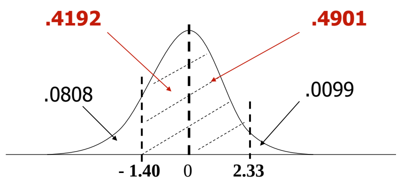
P (–3.00 < *Z* < 3.00) = .9974

z	0.00	0.01
0.0	0.5000	0.504
0.1	0.5398	0.543
0.2	0.5793	0.583
0.3	0.6179	0.621
...		
2.5	0.9938	0.994
2.6	0.9953	0.995
2.7	0.9965	0.996
2.8	0.9974	0.997
2.9	0.9981	0.998
3.0	0.9987	0.998



27

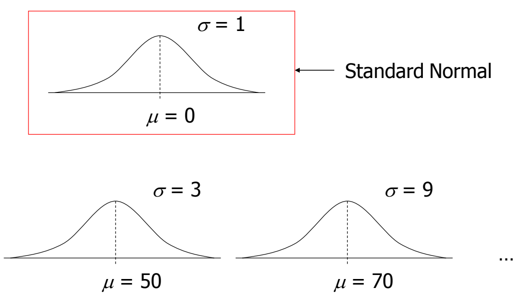
P (–1.40 < *Z* < 2.33) = .9093



Or, P(Z<2.33) – P(Z< – 1.4) = .9901 – .0808 = 9093

28

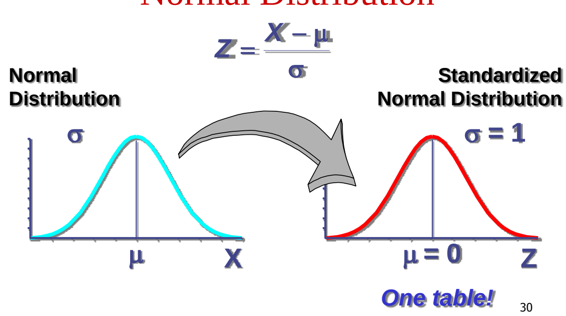
Normal Distribution



29

Standardize the Normal Distribution

$$Z = \frac{X - \mu}{\sigma}$$



30

Theorem

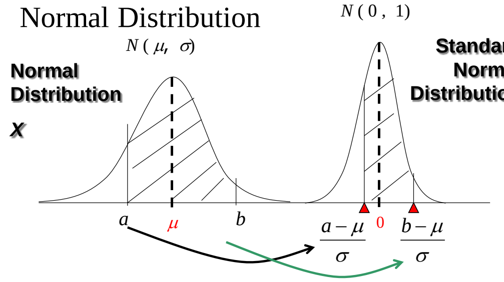
If X is $N(\mu, \sigma^2)$, then $Z = \dfrac{(X - \mu)}{\sigma}$ is $N(0,1)$.

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

31

Standardize the Normal Distribution

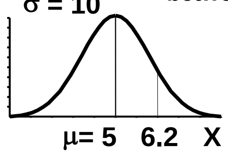


$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

32

Example 1

For a normal distribution that has a mean = 5 and s.d. = 10, what percentage of the distribution is between 5 and 6.2?



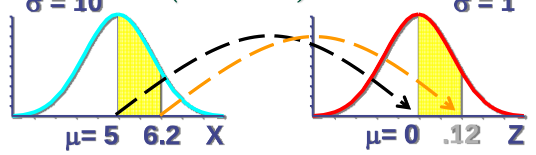
33

Example 1

$$Z = \frac{X-\mu}{\sigma} = \frac{5-5}{10} = 0$$

$$Z = \frac{X-\mu}{\sigma} = \frac{6.2-5}{10} = .12$$

$P(5 \le X \le 6.2)$
$= P(0 \le Z \le .12)$

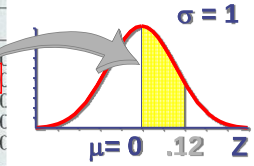


34

Example 1

Standardized Normal Distribution Table

z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
0.3	0.6179	0.6217	0.6255
0.4	0.6554	0.6591	0.6628
0.5	0.6915	0.6950	0.6985



Area = .5478 - .5 = **.0478**

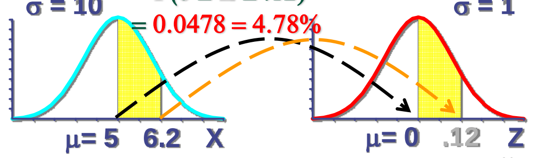
35

Example 1

$$Z = \frac{X-\mu}{\sigma} = \frac{5-5}{10} = 0$$

$$Z = \frac{X-\mu}{\sigma} = \frac{6.2-5}{10} = .12$$

$P(5 \le X \le 6.2)$
$= P(0 \le Z \le .12)$
$= 0.0478 = 4.78\%$



36