Continuous Distributions

Special Probability Densities

The most common parameters in most of the distributions are the lower moments: mean \( \mu \) and variance \( \sigma^2 \).

**Uniform Distribution**

**Definition 6.1:** The continuous random variable \( X \) has a uniform distribution if its p.d.f. is equal to a constant on its support. If the support is the interval \([\alpha, \beta]\), then its p.d.f. is

\[
f(x) = \frac{1}{\beta - \alpha}, \quad \alpha \leq x \leq \beta.
\]

It is usually denoted as \( U(\alpha, \beta) \).

**Example:** Let \( X \) be \( U(0,10) \), find the mean and the variance, and the m.g.f. of \( X \).

\[
\begin{align*}
\mu &= \frac{0 + 10}{2} = 5, \\
\sigma^2 &= \frac{(10 - 0)^2}{12} = \frac{25}{3}, \\
M(t) &= \begin{cases} 
\frac{e^{10t} - 1}{10t}, & t \neq 0, \\
1, & t = 0.
\end{cases}
\end{align*}
\]

Pseudo-Random Number Generator on most computers \( U(0,1) \)

**Gamma Distribution**

**Definition 6.2:** The continuous random variable \( X \) has a Gamma distribution, \( \text{Gamma}(\alpha, \beta) \), if its p.d.f. is

\[
f(x) = \begin{cases} 
\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, & \text{for } 0 \leq x, \\
0, & \text{elsewhere}.
\end{cases}
\]

Gamma Function: \( \Gamma(n) = (n-1)! \)

* \( X \) can be the waiting time until \( \alpha \)-th success in a Poisson process.

**Graphs of Gamma Distributions**

\( \alpha = 4 \)

\( \beta = 4 \)

\( \alpha = 4 \)

\( \beta = 5/6 \)

\( \alpha = 3 \)

\( \beta = 3 \)

\( \alpha = 2 \)

\( \beta = 2 \)

\( \alpha = 1 \)

\( \beta = 1 \)
Continuous Distributions

Gamma Distribution
The mean, variance, and m.g.f. of a continuous random variable \( X \) that has a Gamma distribution are:

\[
\mu = ab, \quad \sigma^2 = ab^2, \quad M(t) = \frac{1}{(1 - \beta t)^a}, \quad t < \frac{1}{\beta}
\]

- Gamma(1, \( \beta \)) \(\Rightarrow\) Exponential Distribution
- Gamma(2, \( r/2 \)) \(\Rightarrow\) Ch-square with d.f. = \( r \).

Special Notation \( \chi^2_{\alpha}(r) \)
Let \( X \) be a random that has Chi-square distribution with degrees of freedom \( r \),

\[
P[X \geq \chi^2_{\alpha}(r)] = \alpha
\]

\[
P[X \leq \chi^2_{1-\alpha}(r)] = \alpha
\]

Example: Find \( \chi^2_{0.05}(3) = 7.815 \)

Try This!
- Find \( \chi^2_{0.05}(7) = ? \)
- Find \( \chi^2_{0.025}(4) = ? \)
- Find the 10th percentile from a \( \chi^2 \) distribution with degrees of freedom 6.

Exponential Distribution
Definition 6.3 The continuous random variable \( X \) has an exponential distribution if its p.d.f. is

\[
f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{for } 0 < x \\ 0, & \text{elsewhere} \end{cases}
\]

where \( \theta \) is the mean of the distribution.

* \( X \) can be the waiting time until next success in a Poisson process.
Continuous Distributions

Exponential Distribution

**Example:** Let \( X \) have an exponential distribution with a mean of 30, what is the first quartile of this distribution?

\[
F(x) = \begin{cases} 
0, & -\infty < x < 0 \\
1 - e^{-x/30}, & 0 \leq x < \infty 
\end{cases}
\]

\( F(\pi_{25}) = 1 - e^{-\pi_{25}/30} = .25 \Rightarrow e^{-\pi_{25}/30} = .75 \)

\( \Rightarrow -\pi_{25}/\theta = \ln(.75) \Rightarrow \pi_{25} = -\theta \cdot \ln(.75) \)

\( \{ \theta = 30 \} \Rightarrow \pi_{25} = -30 \cdot \ln(.75) \Rightarrow \pi_{25} = 8.63 \)

\[ \pi_p = -\theta \cdot \ln(1 - p) \]

Exponential Distribution

Let \( W \) be the waiting time until next success in a Poisson process in which the average number of success in unit interval is \( \lambda \), then, for \( w \geq 0 \)

\[
F(w) = P(W \leq w) = 1 - P(W > w) = 1 - P(W > w)
\]

\( = 1 - P(\text{no success in } [0,w]) = 1 - e^{-\lambda w} \Rightarrow \text{d.f. of exponential distribution.} \)

\[
F'(w) = f(w) = \lambda e^{-\lambda w} \Rightarrow \text{p.d.f. of exponential distribution.}
\]

Exponential Distribution

Suppose that number of arrivals of customers follows a Poisson process with a mean of 10 per hour. What is the probability that the next customer will arrive within 15 minutes? (15 min. = 0.25 hour)

\[
P(X \leq x) = F(x) = 1 - e^{-x/\theta}
\]

\[
P(X \leq 0.25) = F(0.25) = 1 - e^{0.25/\theta}
\]

\( \lambda = 10 \Rightarrow \theta = 0.1 \)

\[
F(0.25) = 1 - e^{-0.25/0.1} = .918
\]

Beta Distribution

**Definition 6.5:** The continuous random variable \( X \) has a **Beta distribution**, \( \text{Beta}(\alpha, \beta) \), if its p.d.f. is

\[
f(x) = \begin{cases} 
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & \text{for } 0 < x < 1 \\
0, & \text{elsewhere.}
\end{cases}
\]

for \( \alpha > 0 \) and \( \beta > 0 \).

**Beta function:**

\[
\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx
\]

Beta Distribution

The mean and variance of a continuous random variable \( X \) that has a **Beta distribution** are:

\[
\mu = \frac{\alpha}{\alpha + \beta}, \quad \sigma^2 = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}
\]
Normal Probability Density Function

**Definition 6.6**: The continuous random variable $X$ has a normal distribution if its p.d.f. is

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$-\infty < x < \infty$

Notation: $\mathcal{N}(\mu, \sigma^2)$ \xrightarrow{} A normal distribution with mean $\mu$ and standard deviation $\sigma$

---

Normal Distribution

The mean, variance, and m.g.f. of a continuous random variable $X$ that has a normal distribution are:

$$E[X] = \mu, \quad \text{Var}[X] = \sigma^2, \quad M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

---

m.g.f. of Normal Distribution

Example: If the m.g.f. of a random variable $X$ is $M(t) = \exp(2t + 16t^2)$, what is the distribution of this random variable? What are the mean and standard deviation of this distribution and what is the p.d.f. of this random variable?

Distribution \xrightarrow{} ?
Mean \xrightarrow{} ?
Standard deviation \xrightarrow{} ?
p.d.f. \xrightarrow{} ?

---

Example

$\mathcal{N}(72, 5) \xrightarrow{}$ A normal distribution with mean 72 and variance 5.
Possible situations: Test scores, pulse rates, ...

$\mathcal{N}(130, 24) \xrightarrow{}$ A normal distribution with mean 130 and variance 24.
Possible situations: Weight, Cholesterol levels, ...

---

Effect of Varying Parameters ($\mu$ & $\sigma$)

$f(x)$
Continuous Distributions

Infinite Number of Tables

Normal distributions differ by mean & standard deviation. Each distribution would require its own table.

That’s an infinite number!

Standard Normal Distribution

Definition 6.7: A normal distribution with \( \mu = 0 \) and \( \sigma = 1 \), is referred to as the Standard Normal Distribution.

Notation:

\[ Z \sim N(\mu = 0, \sigma^2 = 1) \]

Area under Standard Normal Curve

How to find the proportion of the area under the standard normal curve below \( z \) or say \( P(Z < z) = ? \)

Use Standard Normal Table!!!

Standard Normal Distribution

\[ P(Z < 0.32) = \phi(0.32) = \text{Area below .32} = 0.6255 \]

Table for: The Normal Distribution

<table>
<thead>
<tr>
<th>( z )</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.5000</td>
<td>0.5040</td>
<td>0.5080</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5398</td>
<td>0.5438</td>
<td>0.5478</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5793</td>
<td>0.5832</td>
<td>0.5871</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6179</td>
<td>0.6217</td>
<td>0.6255</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6554</td>
<td>0.6591</td>
<td>0.6628</td>
</tr>
</tbody>
</table>

Standard Normal Distribution

\[ P(Z > 0.32) = \text{Area above .32} = 1 - 0.6255 = 0.3745 \]

Areas in the upper tail of the standard normal distribution

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</tr>
</tbody>
</table>

CD - 5
Continuous Distributions

### Standard Normal Distribution

\[ P(0 < Z < 0.32) = \text{Area between 0 and .32} = ? \]

\[ = 0.6255 - 0.5 = 0.1255 \]

<table>
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### Normal Distribution

- \( m = 0 \) \( s = 1 \)
- \( m = 50 \) \( s = 3 \)
- \( m = 70 \) \( s = 9 \)

### Additional Information

- \( P(-1.40 < Z < 2.33) = 0.9093 \)
- \( P(-2.00 < Z < 2.00) = 0.9544 \)
- \( P(-3.00 < Z < 3.00) = 0.9974 \)
- \( P(0 < Z < 1.00) = 0.6826 \)

Standardize the Normal Distribution

One table!

Theorem 6.7: If \( X \) has a normal distribution with mean \( \mu \) and variance \( \sigma^2 \), then \( Z = \frac{X - \mu}{\sigma} \) has a standard normal distribution.

\[
P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}z^2} \, dz
\]

Example

For a normal distribution that has a mean = 5 and s.d. = 10, what percentage of the distribution is between 5 and 6.2?

\[
P(5 < X < 6.2) = P(0 < Z < 0.12)
\]

Standardized Normal Distribution Table

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<td>0.4</td>
<td>0.6554</td>
<td>0.6591</td>
<td>0.6628</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6915</td>
<td>0.6950</td>
<td>0.6985</td>
</tr>
</tbody>
</table>

Area = 0.5478 - 0.5 = 0.0478
Continuous Distributions

Example

P(3.8 ≤ X ≤ 5)

Example

P(X > 8)

Example

P(5 ≤ X ≤ 6.2)

Example

P(X > 8)

More on Normal Distribution

The work hours per week for residents in Ohio has a 

normal distribution with μ = 42 hours & σ = 9 hours.

Find the percentage of Ohio residents whose work 

hours are

A. between 42 & 60 hours.

P(42 ≤ X ≤ 60) = ?

B. less than 20 hours.

P(X ≤ 20) = ?

P(42 ≤ X ≤ 60) = ?

Value 8 is the 62nd percentile

47

CD - 8
Continuous Distributions

\[ P(X \leq 20) = ? \]

\[ Z = \frac{X - \mu}{\sigma} = \frac{20 - 42}{9} = -2.44 \]

\[ P(X \leq 20) = P(Z \leq -2.44) = 0.0073 = 0.73\% \]

Finding Z Values for Known Probabilities

What is \( z \) given \( P(Z < .80) \)?

\[ z = .84 \]

Def. \( z_\alpha \) : \( P(Z \geq z_\alpha) = \alpha \); \( P(Z < z_\alpha) = 1 - \alpha \)

Finding X Values for Known Probabilities

Example: The weight of new born infants is normally distributed with a mean 7 lb and standard deviation of 1.2 lb. Find the 80th percentile.

Area to the left of 80th percentile is 0.800. In the table, there is a area value 0.7995 (close to 0.800) corresponding to a \( z \)-score of .84.

80th percentile = 7 + .84 x 1.2 = 8.008 lb

Finding X Values for Known Probabilities

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80th percentile = 7 + .84 x 1.2 = 8.008 lb

More Examples

- The pulse rates for a certain population follow a normal distribution with a mean of 70 per minute and s.d. 5. What percent of this distribution that is in between 60 to 80 per minute?

- The weights of a population follow a normal distribution with a mean 130 and s.d. 10. What percent of this population that is in between 110 and 150 lbs?

6.6 Normal Approximation for Binomial Probability

\[ P(X \geq 3) = ? \]

\[ \mu = np = 3.5 \]

\[ \sigma^2 = np(1-p) = 1.75 \]
Continuous Distributions

Normal Approximation for Binomial Probability

\[ P(X \geq 3) = P(X > 2.5) = P(Z > \frac{2.5 - 3.5}{\sqrt{1.75}}) = P(Z > -0.76) = 0.7764 \]

Binomial Table:
P(X \geq 3) = 0.7734

\[ \mu = np = 3.5 \quad \sigma^2 = np(1-p) = 1.75 \]

6.7 The Bivariate Normal Distribution

**Definition 6.8**: The pair of random variables \( X \) and \( Y \) have a bivariate normal distribution if and only if their joint probability density function is given by

\[
f(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-r^2}} \exp \left[ -\frac{1}{2(1-r^2)} \left( \frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho \frac{(x-\mu_1)(y-\mu_2)}{\sigma_1 \sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right) \right]
\]

for \(-\infty < x < \infty\) and \(-\infty < y < \infty\), where \( \sigma_1 > 0, \sigma_2 > 0, \) and \(-1 < r < 1\).

**Theorem 6.9**: If \( X \) and \( Y \) have a bivariate normal distribution, the conditional density of \( Y \) given \( X = x \) is a normal distribution with

\[
\mu_{Y|x} = \mu_2 + \frac{\sigma_2}{\sigma_1} (x - \mu_1) \quad \sigma_{Y|x}^2 = \sigma_2^2 (1 - r^2)
\]

\[
\mu_{X|y} = \mu_1 + \frac{\sigma_1}{\sigma_2} (y - \mu_2) \quad \sigma_{X|y}^2 = \sigma_1^2 (1 - r^2)
\]

**Theorem 6.10**: If \( X \) and \( Y \) have a bivariate normal distribution, they are independent if and only if \( r = 0 \).

\[ \text{cov}(X, Y) = \rho \sigma_1 \sigma_2 \]

\[ \rho = \frac{\text{cov}(X, Y)}{(\sigma_1 \sigma_2)} \quad \text{is the correlation coefficient.} \]