Comparison of Two Means

- Paired Samples Test
- Two Independent Samples Test

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Paired Samples

- Data are weight changes of humans, tabulated after administration of a drug proposed to result in weight loss. Each weight change (in kg) is the weight after minus the weight before drug administration. A random sample of 12 subjects was selected to participate in this research.

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Paired Samples

Subject	1	2	3	4	5	6	7	8	9	10	11	12
After	55.2	63.6	58.8	77.2	58.5	69.2	59.5	70.0	68.9	74.0	83.9	74.8
Before	55.4	63.9	60.1	78.8	59.2	68.7	59.9	70.0	69.2	73.7	84.9	75.3
Diff.	-0.2	-0.3	-1.3	-1.6	-0.7	0.5	-0.4	0.0	-0.3	0.3	-1.0	-0.5

Diff. = After – Before
(Paired difference)

δ (or μ_d) is population mean of paired differences
δ_0 is population mean of paired differences under H_0

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Paired-Sample *t*-Test

At $\alpha = 0.05$ level of significance, do these data provide sufficient evidence to indicate that the drug will **help reducing weight**?

H_0: $\delta = \delta_0$ v.s. H_a: $\delta < \delta_0$

Hypothesis:

H_0 : $\delta = 0$ v.s. H_A : $\delta < 0$

or **H_0 : $\mu_d = 0$ v.s. H_A : $\mu_d < 0$**

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Paired-Sample *t*-Test

Test statistic:

$$t = \frac{\bar{x}_d - \delta_0}{s_d / \sqrt{n}}$$

$\bar{x}_d$ mean of paired difference,
s_d standard deviation of paired difference,
n number of paired differences (sample size).

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Paired-Sample *t*-Test

Test statistic:
(Assumption: the paired differences are normally distributed.)

$\bar{x}_d = -.4583$, (*average of the differences*)

$s_d = .6171$, (*standard deviation of the differences*)

$$t = \frac{-.4583 - 0}{.6171 / \sqrt{12}} = -2.573, \text{ degrees of freedom} = 12 - 1 = 11.$$

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Paired-Sample *t*-Test

- **Critical value** is $-t_{.05} = -1.796$,
- ***p*-value** < .025 (approximated using *t*-distribution table, since *t* score of 2.201 has a *p*-value of .025) (*t*-Table)

Decision rule: The null hypothesis is rejected if $t \le -t_{.05} = -1.796$, or ***p*-value < .05**.

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Paired-Sample *t*-Test

Conclusion:
Since p-value < 0.025 < 0.05,
(or $t = -2.573 < -t_{0.05} = -1.796$)
the null hypothesis is rejected. There is sufficient evidence to support the alternative hypothesis that the reduction in weight is statistically significant.

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Two Independent Samples

Two Independent Random Samples were taken from both filtered and non-filtered cigarettes produced by a factory. (number of milligrams of tar.)

Filtered: ($n_1 = 9$)

0.9 1.1 1.2 0.8 1.6 0.9 0.7 1.0 0.9

Non-filtered: ($n_2 = 10$)

1.5 0.9 1.6 0.5 1.4 1.9 1.0 1.3 1.2 1.6

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Notations

μ_1, μ_2 : Population means
σ_1, σ_2 : Population standard deviations
$\bar{x}_1, \bar{x}_2$: Sample means
s_1, s_2 : Sample standard deviations
n_1, n_2 : Sample sizes

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Two Population Means

Is there a significant **difference** between **Average Number of Milligrams of Tar in Filtered and Non-Filtered Cigarette**?

- Null Hypothesis: **No difference** **or** $\mu_1 = \mu_2$
  or $\mu_1 - \mu_2 = 0$
- Alternative Hypothesis:
  There is a difference **or** $\mu_1 \ne \mu_2$
  or $\mu_1 - \mu_2 \ne 0$

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Two Population Means

More general hypothesis statements:

- Null Hypothesis: **No difference** $\mu_1 - \mu_2 = D_0$
- Alternative Hypothesis:
  - $\mu_1 - \mu_2 \ne D_0$
  - $\mu_1 - \mu_2 > D_0$
  - $\mu_1 - \mu_2 < D_0$

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Evidence

- Two Independent Random Samples were taken from both filtered and non-filtered cigarettes.

Filtered: (n_1 = 9)

0.9 1.1 1.2 0.8 1.6 0.9 0.7 1.0 0.9

Non-filtered: (n_2 = 10)

1.5 0.9 1.6 0.5 1.4 1.9 1.0 1.3 1.2 1.6

$\bar{x}_1 = 1.0111, \quad s_1 = .2667, \quad \bar{x}_2 = 1.29, \quad s_2 = .4067$

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Test Statistic

A good statistic for understanding $\mu_1 - \mu_2$ (or D_0):
(or estimating)

$$\bar{x}_1 - \bar{x}_2$$

$\bar{x}_1 = 1.0111, \quad s_1 = .2667, \quad \bar{x}_2 = 1.29, \quad s_2 = .4067$

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Sampling Distribution

If sampling from two independent normal distributions each has mean μ_1 and μ_2 , and standard deviation, σ_1 and σ_2, respectively:

$$\bar{x}_1 - \bar{x}_2 \rightarrow N(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}})$$

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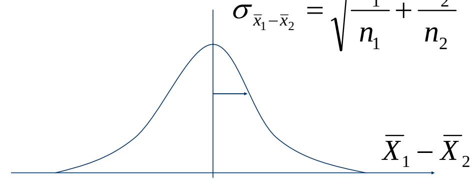
Sampling Distribution

Normal

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$



$$\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$$

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Test Statistic (for Known Variances)

If sampled from normally distributed populations and H_0 is true, i.e., $\mu_1 - \mu_2 = D_0$,

$$z = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \rightarrow \text{Standard Normal Dist.}$$

$$z = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

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Test Statistic (for Large Samples)

If samples are relatively large and H_0 is true, i.e., $\mu_1 - \mu_2 = D_0$,

$$z = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \rightarrow \text{Standard Normal Dist.}$$

$$z = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

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Test Statistic

If sampled from normally distributed populations, and H_0 is true, i.e., $\mu_1 - \mu_2 = D_0$,

$$t = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \rightarrow t - distribution$$

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

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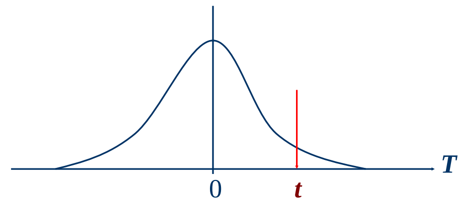
Test Statistic for Decision

Is t statistic extreme?



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Distribution of t -Test Statistic

If $\sigma_1^2 \neq \sigma_2^2$ (Unequal Variances) :

(When $\mathbf{H_0}$: $\mu_1 = \mu_2$, $D_0 = \mu_1 - \mu_2 = 0$)

$$t = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \rightarrow t\text{-distribution}$$

$$\text{with d.f.} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$

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Distribution of t-Test Statistic

If $\sigma_1^2 = \sigma_2^2$ (Equal Variances) :

$$t = \frac{\bar{x}_1 - \bar{x}_2 - D_0}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}} \rightarrow t\text{-distribution}$$

with d.f. = $n_1 + n_2 - 2$

$s_1^2 = s_2^2 = s_p^2$ where $s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$

(When $\mathbf{H_0}$: $\mu_1 = \mu_2$, $\mu_1 - \mu_2 = D_0 = 0$)

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Example

The number of milligrams of tar in filtered and nonfiltered cigarettes are both normally distributed. A research study reported the number of milligrams of tar in **two independent random samples** of cigarettes, 9 filtered cigarettes and 10 non-filtered cigarettes. Data were recorded as the following:

Filtered:	0.9	1.1	1.2	0.8	1.6	0.9	0.7	1.0	0.9	
Nonfiltered:	1.5	0.9	1.6	0.5	1.4	1.9	1.0	1.3	1.2	1.6

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Two Independent Samples

At the **0.05 level of significance**, test whether the average number of milligrams of tar in filtered is significantly **different from** non-filtered cigarettes or not.

Hypothesis:

$\mathbf{H_0}$: $\mu_1 = \mu_2$ **v.s.** $\mathbf{H}_a$: $\mu_1 \neq \mu_2$ (Two-sided test)

or $\mathbf{H_0}$: $D_0 = 0$ **v.s.** $\mathbf{H}_a$: $D_0 \neq 0$

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Two Independent Samples

$\bar{x}_1 = 1.0111, \quad s_1 = .2667, \qquad \bar{x}_2 = 1.29, \qquad s_2 = .4067$

Test statistic: (Assume equal variances.)

$$t = \frac{1.0111 - 1.29}{\sqrt{\frac{.12104}{9} + \frac{.12104}{10}}} = -1.745, \quad \text{d.f.} = 17$$

$$\text{where } s_p^2 = \frac{(9-1)(.2667)^2 + (10-1)(.4067)^2}{9+10-2} = .12104$$

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Two Independent Samples

How do we know if variances are equal?

SPSS Output:

Independent Samples T

		Levene's Test for Equality of Variances				
		F	Sig.	t	df	S
TAR	Equal variances assumed	1.530	.233	-1.745	17	
	Equal variances not assumed			-1.784	15.637	

Equal variances because p-value = .233 > 0.05.

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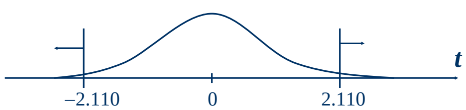
Two Independent Samples

Critical Value Approach:

Reject H_0 if $t \le -t_{0.025,df=17} = -2.110$

or $t \ge t_{0.025,df=17} = 2.110$



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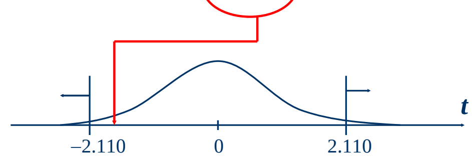
Two Independent Samples

Critical Value Approach:

Critical values

$-t_{0.025,df=17} = -2.110 < t = -1.745 < t_{0.025,df=17} = 2.110$



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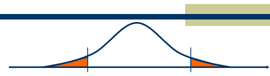
Two Independent Samples

P-value approach:



SPSS Output:

Independent Samples Test

	Levene's Test for Equality of Variances		t-test fo		
	F	Sig.	t	df	Sig. (2-tailed)
ual variances assumed	1.530	.233	-1.745	17	.099
ual variances not assumed			-1.784	15.637	.094

No significant difference because *p*-value = .099 > 0.05.

Two Independent Samples

Conclusion:

- Test of equal variance has an observed *p*-value of .233, hence the **equal variance is assumed**.
- The ***p*-value** is **.099** (see SPSS output), for testing difference between means, the **null hypothesis is not rejected**.
- There is **no sufficient evidence to support the alternative hypothesis** that there is any difference between the average amount of tar in these two types of cigarettes.

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Unequal Variances?

$\bar{x}_1 = 1.0111, \quad s_1 = .2667, \qquad \bar{x}_2 = 1.29, \quad s_2 = .4067$

Test statistic (for unequal var.) :

$$t = \frac{1.0111 - 1.29 - 0}{\sqrt{\frac{.2667^2}{9} + \frac{.4067^2}{10}}} = -1.784,$$

$$\text{d.f.} = \frac{\left(\frac{.2667^2}{9} + \frac{.4067^2}{10}\right)^2}{\frac{\left(\frac{.2667^2}{9}\right)^2}{8} + \frac{\left(\frac{.4067^2}{10}\right)^2}{9}} = 15.637$$

When using t-table, one can round **off** the d.f. to 15.

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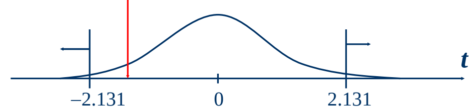
Unequal Variances

Critical Value Approach:

$$-2.131 < t = -1.784 < t_{0.025, df=15} = 2.131$$



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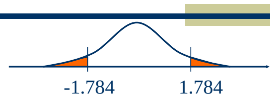
Two Independent Samples

P-value approach:



SPSS Output:

Independent Samples Test

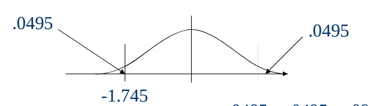
	Levene's Test for Equality of Variances F	Sig.	t-test fo... t	df	Sig. (2-tailed)
ual variances assumed	1.530	.233	-1.745	17	.099
ual variances not assumed			-1.784	15.637	.094

No significant difference because *p*-value = .094 > 0.05.

One-sided Test

What is the *p*-value if the goal is to test whether the filtered cigarettes **have less amount of tar** than the non-filtered cigarettes?

$H_0: \mu_1 = \mu_2 \quad \text{v.s.} \quad H_a: \mu_1 < \mu_2$



***p*-value = .0495**

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Confidence Interval

The $(1-\alpha)$% confidence interval estimate for the difference of the two population means $\mu_1 - \mu_2$ is

If $\sigma_1^2 \neq \sigma_2^2$, $\quad \bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

If $\sigma_1^2 = \sigma_2^2$, $\quad \bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2} \cdot \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$

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When to use t-Test for comparing two populations?

Condition 1: Unknown population variances.

Condition 2:

- Sampling from normal populations
  OR
- Having relatively large samples

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