Discrete Probability Distributions

- Probability Distributions
- Mean and Expectation
- Binomial Distribution

What is the probability distribution of rolling a die?

If outcomes are equally likely to occur, the distribution is

$P(1) = 1/6$, $P(2) = 1/6$,
$P(3) = 1/6$, $P(4) = 1/6$,
$P(5) = 1/6$, $P(6) = 1/6$,

and total probability is 1.

x_i	$P(x_i)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Review

Probability Distribution Model

	x_i	$P(x_i)$	
x_1 →	1	1/6	← $P(x_1)$
x_2 →	2	1/6	← $P(x_2)$
x_3 →	3	1/6	← $P(x_3)$
x_4 →	4	1/6	← $P(x_4)$
x_5 →	5	1/6	← $P(x_5)$
x_6 →	6	1/6	← $P(x_6)$

Review

Relative Frequency and Probability Distribution (Model)

Number of times visited doctor last year from a sample of 300 individuals

	Class	Frequency	Relative Frequency	
x_1 →	**0**	**54**	**.18**	← $P(x_1)$
x_2 →	**1**	**117**	**.39**	← $P(x_2)$
x_3 →	**2**	**72**	**.24**	← $P(x_3)$
x_4 →	**3**	**42**	**.14**	← $P(x_4)$
x_5 →	**4**	**12**	**.04**	← $P(x_5)$
x_6 →	**5**	**3**	**.01**	← $P(x_6)$
	Total	**300**	**1.00**	

Review

Expected Value

Empirical study: Play the game 1000 times, if head turns you win $1, otherwise, you win $0. On average, how much money do you win per game?

	Outcome	Frequency	Relative Frequency
x_1	Head($1)	500	.5 $P(x_1)$
x_2	Tail ($0)	500	.5 $P(x_2)$
	Total	1000	1

Expected Value

Empirical study: Play the game 1000 times, if head turns you win $1, otherwise, you win $0. On average, how much money do you win per game?

Average $= (1\text{x}500 + 0\text{x}500) / 1000$

$= 500 / 1000 = .5$

or $= 1\text{x}\overset{P(x_1)}{\boxed{500/1000}} + 0\text{x}\overset{P(x_2)}{\boxed{500/1000}}$

$= \underset{x_1}{1}\text{x}.5 + \underset{x_2}{0}\text{x}.5 = .5$

Expected Value

Empirical study: Play the game 1000 times, if head turns you win $1, otherwise, you win $0. On average, how much money do you win per game?

	Outcome, x_i	Relative Frequency, $P(x_i)$	Product, $x_iP(x_i)$
x_1	Head($**1**)	.5 $P(x_1)$	1 x .5 $x_1\ P(x_1)$
x_2	Tail ($**0**)	.5 $P(x_2)$	0 x .5 $x_2\ P(x_2)$
	Total	1.0	**.5** $\Sigma\ x\ P(x)$

Measure of Center for a Distribution

Suppose that all the possible outcomes in a sample space of a random experiment are $x_1, x_2, \ldots, x_k$, and that $P(x_i)$ is the probability of outcome x_i. The mean, μ, of this probability model is

$$\mu = x_1\,P(x_1) + x_2\,P(x_2) + \ldots + x_k\,P(x_k) = \Sigma\ x\ P(x)$$

Outcomes(x_i)	x_1	x_2	…	x_k
Probabilities(P)	$P(x_1)$	$P(x_2)$	…	$P(x_k)$

Measure of Center for a Distribution

$$\mu = x_1\,P(x_1) + x_2\,P(x_2) + \ldots + x_k\,P(x_k)$$

Example: Toss a balanced coin and interested in number of heads turn up.

{x_1 = 1 means “Head” and x_2 = 0 means “not Head”, and $P(x_1)$ = .5, $P(x_2)$ = .5 are there corresponding probabilities.}

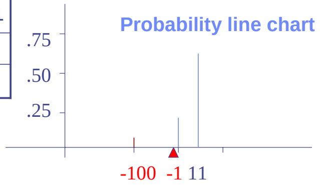
Outcomes, x_i	Probabilities, $P\ (x_i)$	Product
1	.5	1 x .5
0	.5	0 x .5

So, $\mu = 1 \cdot P(x_1) + 0 \cdot P(x_2) = 1 \cdot .5 + 0 \cdot .5 =$ **.5**

Is it a profitable insurance premium?

Premium(x_i)	-100	-1	11
Probability(P)	.1	.25	.65

x_i	$P(x_i)$	$x_iP(x_i)$
-100	.1	-100x.1
-1	.25	-1x.25
11	.65	11x.65



The mean of the distribution is

$$\mu = (-100)\ x.1 + (-1)x.25 + 11x.65 = -3.1$$

(Weighted by probabilities.)

Example

Toss a coin: When the outcome is Head, your net gain is $3.00. If the outcome is tail, your net loss is $2.00. If the coin is unbalanced, and the probability of head is .3, is this a fair game? How much money do you expect to lose or win per game?

x_i	$P(x_i)$	$x_i\ P(x_i)$
3	.3	3x.3
–2	.7	–2x.7

μ = 3x.3 + (-2)x.7 = -.5; expected to lose $.5 per game.

Game

A Game: Roll a die. Pay $2.00 for each game. When outcome is 6, your net gain is $3.00. If outcome is 4 or 5, your net gain is $1.00. If other outcomes occur, you lose your bet. Is this a fair game? How much money do you expect to win per game for player? What is the variance of the outcomes in this game?

Measure of Spread

Suppose that all the possible outcomes in a sample space of a random experiment is $x_1, x_2, \ldots, x_k$, and that $P(x_i)$ is the probability of outcome x_i. The variance, σ^2, of this probability model is

$$\sigma^2 = (x_1-\mu)^2 P(x_1) + (x_2-\mu)^2 P(x_2) + \ldots + (x_k-\mu)^2 P(x_k)$$
$$= \Sigma\, (x-\mu)^2\, P(x)$$

Outcomes(x_i)	x_1	x_2	…	x_k
Probabilities(P)	$P(x_1)$	$P(x_2)$	…	$P(x_k)$

Measure of Spread

Example: Toss a balanced coin and interested in number of heads turn up. {x_1 = 1 implies "Head" and x_2 = 0 implies "not Head", and $P(x_1)$ = .5, $P(x_2)$ = .5 are there corresponding probabilities.}

μ = .5

x_i	$P(x_i)$	$x_i-\mu$	$(x_i-\mu)^2$	$(x_i-\mu)^2 P(x_i)$
1	.5	1–.5	.25	.25 x .5
0	.5	0–.5	.25	.25 x .5

SUM = .25

So, σ^2 = .25 ·.5 · + .25 ·.5 = **.25**

Binomial Probability

The probability of getting two 6's in roll a balanced die 5 times experiment.

$P(S\cap S\cap S'\cap S'\cap S') = (1/6)^2 \times (5/6)^3$

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$P(S\cap S'\cap S'\cap S\cap S') = (1/6)^2 \times (5/6)^3$

…

How many of them? $\binom{5}{2} = \dfrac{5!}{2!\cdot 3!} = 10$

Binomial Probability

The probability of getting two 6's in roll a balanced die 5 times experiment is

$P(2\ S\text{'s}) = \binom{5}{2} \times (1/6)^2 \times (5/6)^3$

$= 5!/(2!3!) \times (1/6)^2 \times (1 - 1/6)^3$

Binomial Probability Model

In a binomial experiment involving ***n*** independent and identical Bernoulli trials each with probability of success ***p***, the probability of having ***x*** successes can be calculated with the ***binomial probability mass function***, and it is, for x = 0, 1, …, n,

$$\boxed{P(X = x) = \frac{n!}{x!\cdot(n-x)!}\cdot p^x \cdot (1-p)^{n-x}}$$
$$= \binom{n}{x}\cdot p^x \cdot (1-p)^{n-x}$$

Parameters of Binomial Distribution

Parameters of the distribution:

Mean of the distribution, $\mu = n\cdot p$

Variance of the distribution, $\sigma^2 = n\cdot p\cdot(1 - p)$

Standard deviation, σ , is the **square root of variance**.

Binomial Probability

Example: If there are 10% of the population in a community have a certain disease, what is the probability that 4 people in a random sample of 5 people from this community has the disease? (Assume binomial experiment.)

Identify n = 5, x = 4, p = .10

$P(X=4) = \frac{\mathbf{5!}}{\mathbf{4!\ 1!}} \cdot (.10)^4 \cdot (1 - .10)^{5-4}$

$= 5 \cdot (.10)^4 (.90)^1$

$= .00045$

$$P(X = x) = \frac{n!}{x!\cdot(n-x)!} \cdot p^x \cdot (1-p)^{n-x}$$

Binomial Probability

Example: In the previous problem, what is the probability that 4 or more people have the disease?

Identify n = 5, x = 4, p = .10

$P(X \geq 4) = P(X=4) + P(X=5)$

$= .00045 + \frac{\mathbf{5!}}{\mathbf{5!\ 0!}} \cdot (.10)^5 \cdot (1 - .10)^{5-5}$

$= .00045 + .00001 = .00046$

(What is this number telling us?)

Binomial Distribution

n = 5, p = .10

P(0) = .5905

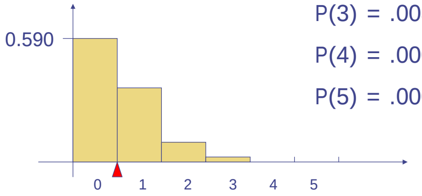
P(1) = .3281

P(2) = .0729

P(3) = .0081

P(4) = .0004

P(5) = .00001



Example: If there are 10% of the population in a community have a certain disease, what is the probability distribution for the number of people having this disease in a random sample of 5 people selected from this community?

n = 5, p = .10

μ = 5 x .10 = .5

σ^2 = 5 x .1 x (1–.1) = .45

P(0) = .5905

P(1) = .3281

P(2) = .0729

P(3) = .0081

P(4) = .0004

P(5) = .00001